STUDY OF QUEUES OF PRIORITY MESSAGES IN A SINGLE CHANNEL

Sam Aaron Dressin

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SAM A. DRESSIN

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by

Sam Aaron Dressin

Major, U.S.Marine Corps

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PREFACE

Having served a number of years in the capacity of communications officer in U.S. Marine Corps Infantry Divisions, the writer has been aware of the need for a method of analysing military communication systems. The delay of messages has been and still is of major concern to all military communicators. The increasing speed of action of military units ages information rapidly; and as a result message delays achieve greater significance.

It was thus with great pleasure and interest that the writer undertook the problem of investigating military message delay times as recommended by Dr. Edgar Reich of The Rand Corporation. The major part of the work was accomplished at The Rand Corporation, Santa Monica, California during the industrial tour phase of the Electronics Curriculum as presented at the U.S. Naval Postgraduate School, Monterey, California.

The writer takes pleasure in acknowledging his indebtedness to Dr. Edgar Reich, whose assistance, encouragement and instruction in queueing theory were prerequisites for the preparation of this work.

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TABLE OF SYMBOLS AND ABBREVIATIONS

$H(t)$	- Message length distribution, probability that message length is less than t.
$h(t)$	- $\frac{d}{dt} H(t)$
$\frac{1}{\mu}$	- mean of message length distibution.
$f_n(t)$	- probability that n messages arrive within time interval t.
λ	- mean arrival rate.
Channel State n	- state of channel when n messages are in the system.
$P_n(t)$	- probability that at time t, the channel is in state n.
$W_n(t)$	- conditional probability that the delay will be t or less for a message arriving when channel state is n.
$w_n(t)$	- $\frac{d}{dt} W_n(t)$
$W(t)$	- probability that the delay will be t or less.
$w(t)$	- $\frac{d}{dt} W(t)$
λ_p	- average arrival rate of priority p.
$\lambda_{\Sigma p}$	- $\sum_{1}^{p} \lambda_i$
Channel State n,p	- when $n > 0$ one message of any priority being transmitted, and n - 1 messages of priority p or higher awaiting service.

Channel State Zero	- no messages being transmitted and none awaiting service.
$P_{n,p}(t)$	- probability that at time t, the channel is in state n,p.
$W_{n,p}(t)$	- conditional probability that the delay will be t seconds or less for a priority p message arriving when channel state is n,p.
$w_{n,p}(t)$	- $\frac{d}{dt} W_{n,p}(t)$
$W_p(t)$	- probability that the delay for a priority p message will be t or less.
$w_p(t)$	- $\frac{d}{dt} W_p(t)$
α	- $\frac{\lambda_{\Sigma p-1}}{\lambda}$
β	- $\frac{\lambda_{\Sigma p}}{\lambda}$
ρ	- $\frac{\lambda}{\mu}$
ν_i	- i th central moment

CHAPTER I

INTRODUCTION

The problem of formulating a systematic method for the determination of communication requirements of Army combat units is being investigated by Haller, Raymond and Brown, Inc. [1]. In determining requirements, it is first necessary to establish relationships between the communication load and message delay times for various communication systems. Commercial communication organizations have in the past applied queueing theory to the solution of similar problems [2]. However, since the communication load and discipline of the military are different from those of civilian users, the presently available tools of queueing theory are not directly applicable to military problems. Research in two directions is thus required. It is necessary to determine more accurately the load characteristics of military communications and to further develop queueing theory with military applications in mind. This paper discusses some of the queueing theory techniques available and develops the theory a step further.

Queueing theory is the application of the mathematical theory of probability to problems involving traffic congestion [3]. The traffic congestion occurs when a service facility is subjected to service demands which vary statistically. The traffic may be aircraft attempting to land at a airport, customers at a bank teller's window, or messages in a communication system. We assume that there are significant statistical fluctuations of either the inter-arrival times,

or the service times, or both; and that it is undesirable for the traffic to wait. An analysis of the waiting lines or queues that result may lead to methods for shortening the lengths of the lines and thereby decreasing the delays encountered by the traffic.

Chapters II and III contain the necessary groundwork for an understanding of the application of queueing theory to communication problems.

Chapter IV contains the solution to a classical queueing problem.

Chapter V contains the major contribution of this work, consisting of the solution to the queueing problem characterized by the following:

a. single channel

b. messages from an exponential length distribution

c. messages arriving in accordance with Poisson Laws

d. messages transmitted in accordance with assigned priorities.

The solution consists of mathematical expressions and graphs from which the communicator is able to compute the probabilities that messages of the various priorities will be subjected to particular delays. It is thus possible for him to determine how well the channel will perform and perhaps make any necessary changes to correct undesirable delays.

CHAPTER II

PARAMETERS OF THE COMMUNICATIONS CHANNEL

1. Introduction

The over-all problem of military communications involves the exchange of information among a number of sources. The amount of information and the number of sources depend on the size of the unit and the tactical situation. The solution of the problem would result in determining the number of channels to be established and communication procedures to be enforced. It appears then, that the solution to the over-all problem would require an extensive knowledge of the influence of various parameters on the operation of a single channel. Single channel as used here-in applies to a standard military radio net.

In order to treat the operation of a channel mathematically, it is necessary to specify the following:

1. Message generation times
2. Message length distribution
3. Queue discipline

As noted earlier, either the message generation times or the message lengths, or both, fluctuate statistically. The application of queueing theory may then result in the calculation of the probability that a message will wait a time t between it's origination by the sender and the beginning of it's transmission over the channel. This probability is termed the delay or waiting time probability. It should be noted that the time t does not include the transmission time of the

message but is the time the message waits until it's transmission begins.

2. Message Generation Times

Message generation times refer to the intervals of time between the origination of messages by the units utilizing the channel. These time intervals may be described by stating the probabilities of having various numbers of messages generated per unit of time; or the probabilities that intervals between message arrivals will be of various lengths. The message generation times for military units will vary depending on the size of the unit, the mission, and numerous other factors. Haller, Raymond and Brown, Inc. is attempting to determine empirically the message generation distribution for company-size units. There are undoubtedly dependencies between messages since incoming messages prompt outgoing messages. However, in large units it is expected that these dependencies average out.

3. Message Lengths

Message lengths may be described by stating the probability that a message chosen at random will have a length less than a given amount. The message length distribution may also be influenced by the tactical situation.

4. Queue Discipline

The term queue discipline refers to the handling of messages awaiting transmission. Military communications procedure usually requires that delayed calls be serviced consecutively in the order of their origination unless a priority system is in effect.

CHAPTER III

PROBABILITY DISTRIBUTIONS OF THE PARAMETERS

1. Introduction

The exact probability distributions of message generation times and message lengths of military units have not as yet been determined. However, if the discussion is limited to units of about division size during periods of greatest activity, it may be assumed that the message length distribution is exponential and the message generation times distribution (number of messages generated per unit time) is Poisson. These are considered the best assumptions until empirical data demonstrates otherwise.

2. Exponential Message Lengths

Suppose that a message is being transmitted, then message length and transmission time may be used synonomously. Let the length of transmission , in seconds, be represented by a random variable t; possessing a probability distribution. The transmission may end during any small interval of time. If the probability of termination during any small time interval is constant and independent of how long the transmission has been in progress, the message length distribution is exponential. Telephone company data demonstrates that the exponential distribution is an accurate portrayal of telephone conversation lengths [4].

The exponential distribution may be derived from the assumption that the probability of a message having a length between t and t + h

(h is a short time interval) is constant and independent of t [4] [5]. Here we will accept this distribution function and investigate its characteristics.

If a message is chosen at random from an exponential distribution, the probability that its length is less than t is

$$H(t) = 1 - \epsilon^{-\mu t} \qquad t \geq 0$$

The message length probability density function is

$$\frac{d\,H(t)}{dt} = h(t) = \mu \epsilon^{-\mu t} \qquad t \geq 0$$

The average message length is

$$\int_0^\infty t\,h(t)\,dt = \int_0^\infty t \mu \epsilon^{-\mu t} dt = \frac{1}{\mu}$$

Let zero time represent the time at which transmission of a randomly chosen message begins. Let t represent a time of observation at or subsequent to time zero. Then to demonstrate that the probability that the transmission will end during h is independent of t, write the probability that a message be of length between t and t + h

$$H(t + h) - H(t)$$

Let P = conditional probability that if a transmission has been in progress for a time t, the message length is between t and t + h.

Then by Bayes' Theorem

$$P = \frac{H(t + h) - H(t)}{1 - H(t)}$$

$$= \frac{\left[1 - \epsilon^{-(t + h)}\right] - \left[1 - \epsilon^{-\mu t}\right]}{1 - \left[1 - \epsilon^{-\mu t}\right]}$$

$$= 1 - \epsilon^{-\mu h}$$

Thus if a transmission has been in progress for a time t, the probability

that it will end during the following interval of time h is

$$1 - \epsilon^{-\mu h}$$

and is independent of t.

If h is assumed to be small, then by expanding the exponential and discarding terms in h^2 and higher powers

$$1 - \epsilon^{-\mu h} \approx \mu h.$$

3. Poisson Message Generation Times

If messages are generated independently of each other, and if the probability that a message is generated in a time interval is independent of the number of messages generated in previous time intervals, the message generation process is considered to be Poisson. The Poisson distribution may be derived by postulating that the probability of an event occuring in a time interval is a constant and independent of the number of events that occurred previously [4] [5]. Here we accept this function and investigate its properties.

For Poisson arrivals the probability that n arrivals occur during a time interval t is

$$f_n(t) = \frac{(\lambda t)^n \epsilon^{-\lambda t}}{n!}$$

The average number of arrivals per unit time is

$$\frac{\bar{n}}{t} = \sum_{n=o}^{\infty} \frac{n f_n(t)}{t} = \sum_{n=o}^{\infty} \frac{n(\lambda t)^n \epsilon^{-\lambda t}}{t n!} = \lambda$$

If h represents a small time interval, then

$$f_o(h) = \epsilon^{-\lambda h} \approx 1 - \lambda h$$

is the probability that there will be no arrivals during time h; and

$$f_1(h) = \lambda h \, \epsilon^{-\lambda h} \approx \lambda h$$

is the probability that there will be one arrival during h. All probabilities of more than 1 arrival consist of terms of order h^2 and higher and are discarded.

CHAPTER IV

EXPONENTIAL MESSAGE LENGTHS, POISSON ARRIVALS AND ORDERED QUEUEING IN A SINGLE CHANNEL

1. Introduction

Assume that the population of messages to be transmitted possesses an exponential length distribution, the message generation process is Poisson, and that the messages are serviced on a "first come, first served" basis. The information desired is the distribution function of the waiting time. The attack on the problem consists of determining the probability that the channel is in a particular state and the conditional probability that a message will wait less than time t if arriving while the channel is in a particular state. The summation of the products of these two probabilities will result in the probability that a message will wait a time less than t before its service begins.

2. State Probabilities

Define system state n to be the state of the channel when there are n messages in the system being served or awaiting service. $P_n(t)$ is used to designate the probability that at a time t the system state is n. If h represents a small interval of time, then from the discussion on distributions, μh represents the probability that a transmission will terminate during h and λh represents the probability that a message will arrive during h.

List now the possible ways in which the system can arrive at state n, $(n \geq 1)$, at time t:

1. at t - h the state is n, no arrivals and no transmission completions during h.

2. at t - h the state is n+1, no arrivals and one transmission completion during h.

3. at t - h the state is n - 1, one arrival and no transmission completions during h.

4. all others (states other than n, n-1, n+1, and multiple arrivals and/or completions).

The probabilities of these events are:

1. $(1-\mu h)(1-\lambda h)P_n(t-h) \approx [1-(\mu+\lambda)h]\ P_n(t-h)$

2. $(\mu h)(1-\lambda h)P_{n+1}(t-h) \approx \mu h P_{n+1}(t-h)$

3. $(\lambda h)(1-\mu h)P_{n-1}(t-h) \approx \lambda h P_{n-1}(t-h)$

4. order of h^2 and higher

The sum of these is the probability that state n will be reached at time t.

$$P_n(t)= [1-(\mu+\lambda)h]\ P_n(t-h)+\mu h P_{n+1}(t-h)+\lambda h P_{n-1}(t-h)$$

$$\frac{P_n(t)-P_n(t-h)}{h}= -(\lambda+\mu)P_n(t-h)+\mu P_{n+1}(t-h)+\lambda P_{n-1}(t-h)$$

letting $h \to 0$

$$(1)\quad \frac{dP_n(t)}{dt}= -(\mu+\lambda)P_n(t)+\mu P_{n+1}(t)+\lambda P_{n-1}(t) \qquad n \geq 1$$

Now consider state zero:

1. at t-h state is zero, no arrivals and no transmissions completed during h

2. at t-h state is 1, no arrivals and one transmission completed during h

3. all others

The probabilities are:

1. $(1-\mu h)(1-\lambda h)P_0(t-h) \approx [1-(\mu+\lambda)h] \; P_0(t-h)$

2. $(\mu h)(1-\lambda h)P_1(t-h) \approx \mu h P_1(t-h)$

3. order of h^2 and higher

$$P_0(t) = [1-(\mu+\lambda)h] \; P_0(t-h) + \mu h P_1(t-h)$$

$$\frac{P_0(t)-P_0(t-h)}{h} = -(\mu+\lambda)P_0(t-h) + \mu P_1(t-h)$$

letting $h \to 0$

$$(2) \quad \frac{dP_0(t)}{dt} = -(\mu+\lambda)P_0(t) + \mu P_1(t)$$

It has been demonstrated [6] that when $\lambda < \mu$ and t is large, that the effect of the initial conditions are overcome and that the derivatives of equations 1 and 2 approach zero. Then

$$(\lambda+\mu)P_n = \mu P_{n+1} + \lambda P_{n-1} \qquad n \geq 1$$

$$\lambda P_0 = \mu P_1$$

Assume a solution of the form $P_n = Ax^n$. Then

$$(\lambda+\mu)x = \mu x^2 + \lambda$$

$$x = \frac{\lambda}{\mu}, x = 1$$

Since $\sum_{n=0}^{\infty} P_n(t) = 1$ for all t, the solution x=1 is discarded,

and $x = \frac{\lambda}{\mu}$, then

$$P_n = A\left(\frac{\lambda}{\mu}\right)^n$$

$$\sum_{n=0}^{\infty} A\left(\frac{\lambda}{\mu}\right)^n = 1 = \frac{A}{1-\frac{\lambda}{\mu}} \qquad \text{or}$$

$$A = 1-\frac{\lambda}{\mu} \qquad \text{and}$$

$$P_n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

3. Waiting Time Distribution

Let $W_n(t)$ represent the conditional probability that if a message arrives when the channel is in state n, its waiting time is less than t. Let $w_n(t)$ be the probability density function. Consider a message M arriving when the channel state is n. There are then n messages each with an exponential length distribution to be transmitted prior to M. The density function of M's waiting time is the convolution of n exponential density functions

$$w_n(t) = \frac{\mu e^{-\mu t}(\mu t)^{n-1}}{(n-1)!}$$

The density function of a message's waiting time when the channel state is unknown is

$$w(t) = \sum_{n=0}^{\infty} P_n w_n(t) = \mu e^{-\mu t}\left(1-\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \frac{(\mu t)^{n-1}}{(n-1)!}$$

$$= \lambda e^{-\mu t}\left(1-\frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} = \lambda\left(1-\frac{\lambda}{\mu}\right) e^{-(\mu-\lambda)t}$$

The distribution function of the waiting time is

$$W(t)=\int w(t)dt$$

$$=\lambda(1-\frac{\lambda}{\mu})\int e^{-(\mu-\lambda)t}dt$$

Applying the condition that $W(0)=(1-\frac{\lambda}{\mu})$, since the probability of waiting zero time is merely the probability that the message arrives when the channel is in state zero,

$$W(t)=1-\frac{\lambda}{\mu}e^{-(\mu-\lambda)t}$$

The mean waiting time may be computed from

$$\bar{t}=\int_0^\infty tw(t)dt$$

$$=\lambda(1-\frac{\lambda}{\mu})\int_0^\infty t\,e^{-(\mu-\lambda)t}$$

$$\bar{t}=\frac{\lambda}{\mu(\mu-\lambda)}$$

CHAPTER V

POISSON ARRIVALS, EXPONENTIAL MESSAGE LENGTHS, AND PRIORITY QUEUEING IN A SINGLE CHANNEL

1. Introduction

The problem solved in Chapter IV is among the simplest in queueing theory [7]. Among the problems that have been solved are:

Single Channel

1. Poisson arrivals, exponential serving time, service in order of arrival.
2. Poisson arrivals, exponential serving time, random service.
3. Poisson arrivals, uniform serving time, service in order of arrival.
4. Poisson arrivals, Pearson III serving time, service in order of arrival.
5. Arrival rate dependent on channel state, Pearson III serving time, service in order of arrival.
6. Poisson arrivals, any serving time distribution, service in order of priorities, solved for mean waiting time only.

Multiple Channel

1. Poisson arrival, exponential serving time, service in order of arrival.
2. Poisson arrivals, exponential serving time, service in order of priorities, solved for mean waiting time only.

The two priority cases mentioned above are of interest to military connumicators. However the solutions exist for mean waiting time only and furnish merely an indication of the delays encountered by the messages. In military operations messages subjected to very long delays are not compensated by messages having very short delays. The military communicator would prefer knowledge of the waiting time distribution function which would enable him to evaluate his communication system completely. In this chapter a waiting time distribution function is derived for a single channel characterized by the following:

1. Poisson arrivals
2. exponential serving time
3. service in order of priorities

2. Assumptions

The following discussion is limited to the case in which the messages arrive independently of each other and the arrival probabilities are independent of the state of the channel. Thus the units arrive in accordance with independent Poisson Laws with λ_p representing the average arrival rate of priority p. It is also assumed that all priorities possess identical exponential message length probability distribution functions.

The number of different priorities is r, with 1 representing the highest and r the lowest priorities. The queue discipline follows standard military communications procedure which results in higher priority messages displacing messages of lower priority in the waiting line (but no disturbance of the message being transmitted), and

messages of equal priority being transmitted in order of arrival.

In principle the attack on the problem consists of:

1. determining the conditional density function $w_{n,p}(t)$
2. determining $P_{n,p}$
3. determining the unconditional density function $w_p(t)$ by

$$\sum_{n=0}^{\infty} P_{n,p} w_{n,p}(t)$$

3. Conditional Waiting Time Distribution

In attempting to determine the conditional probability that a message of priority p will wait t or less if arriving when the channel state is n,p, consider a message M of priority p arriving when the system is in 1,p,i.e., no messages of priority p or higher awaiting service and one message of any priority being transmitted. During the latter message's transmission, messages of priorities higher than p may arrive; and these will be served before M. M will wait a time τ when the transmission of a message will just have been completed and there are no messages of priority higher than p awaiting service. The density function of τ ("busy period") has been determined by Kendall [8] and Bailey [9] , and when applied to the present problem becomes

$$w_{1,p}(\tau) = \left(\frac{\mu}{\lambda_{\Sigma p-1}}\right)^{\frac{1}{2}} \quad \frac{e^{-(\lambda_{\Sigma p-1}+\mu)\tau} I_1(2\tau\sqrt{\lambda_{\Sigma p-1}/\mu})}{\tau}$$

$\lambda_{\Sigma p-1}$ is used here since the only messages that affect M's waiting time are those of priorities higher than p. Since the individual

arrival distributions of these messages are independent Poisson, the sum is also Poisson with mean $\lambda_{\sum p-1}$.

Now assume that a message M of priority p arrives when the channel state is n,p. M must wait a time t_n until the following messages are transmitted:

1. the n messages contributing to state n,p when M arrived.
2. all messages of priorities higher than p arriving before M's service begins.

Consider an identical system in which a message M' of priority p arrives when the channel state is (n-1), p. M and M' arrive at the same time. Let M' wait t_{n-1}, then M must wait an additional time τ. Thus if t_n is M's waiting time:

$$t_n = t_{n-1} + \tau$$

or for n=2, this recursion formula gives

$$t_2 = t_1 + \tau \quad \text{but } t_1 = \tau$$

then the density function of t_2,

$$w_{2,p}(t) = w_{1,p}(t) * w_{1,p}(t)$$

and in general, the density function of the waiting time for a message of priority p arriving when the channel state is n,p

$$w_{n,p}(t) = w_{1,p}(t)^{*n}$$

Transform pair 576.1, Campbell and Foster [10] furnishes the characteristic function of $w_{1,p}(t)$

$$\frac{4\mu}{\left[(s+\gamma)^{\frac{1}{2}}+(s+\sigma)^{\frac{1}{2}}\right]^2}$$

Where
$$\gamma = \lambda_{\sum p-1}+\mu+2\sqrt{\lambda_{\sum p-1}\,\mu}$$

$$\sigma = \lambda_{\sum p-1}+\mu-2\sqrt{\lambda_{\sum p-1}\,\mu}$$

The characteristic function of $w_{n,p}(t)$

$$C_{n,p}= \frac{4\mu^n}{\left[(s+\gamma)^{\frac{1}{2}}+(s+\sigma)^{\frac{1}{2}}\right]^{2n}}$$

and utilizing pair 576.1 again

$$w_{n,p}(t) = \left(\frac{\mu}{\lambda_{\sum p-1}}\right)^{\frac{n}{2}} n e^{-(\lambda_{\sum p-1}+\mu)t}\, \frac{I_n\left(2t\sqrt{\lambda_{\sum p-1}\,\mu}\right)}{t} \qquad n > 0$$

$$w_{0,p}(t)= \delta(t-0)$$

4. Channel State Probabilities

In determining the channel state probabilities, note that the probability of state zero is

$$\left(1-\frac{\lambda}{\mu}\right) \quad \text{where } \lambda = \sum_{1}^{r}\lambda_p \,.$$

State 1,p can exist in the following ways:

1. only one message in the channel of any priority
2. more than one message in the channel all of lower priority than p
3. more than one message in the channel, the one being served of priority p or higher and all others lower than p

The probability of 1,i.e. of the channel containing only one message is

$$(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})$$

The following may occur resulting in the existence of state 1,p at time t:

1. State at time t-h is 1,p, no completions, no arrivals of priorities p or higher, and either arrivals or no arrivals of priorities lower than p during h. The probability of this event is

$$\left[1-(\lambda_{\Sigma p}+\mu)\right]P_{1,p}(t-h)$$

2. State at time t-h is 1,p (less the condition of exactly one message in the channel), and a completion during h. The probability of this event is

$$\mu h\left[P_{1,p}(t-h)-(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})\right]$$

3. State at time t-h is zero and an arrival of any priority

during h. The probability of this event is

$$\lambda h P_0(t-h)=(1-\frac{\lambda}{\mu})\lambda h$$

4. State at time t-h is 2,p and a completion during h. The probability of this event is

$$\mu h P_{2,p}(t-h)$$

Thus:

$$P_{1,p}(t)=P_{1,p}(t-h)\left[1-(\mu+\lambda_{\Sigma p})h\right]+\mu h\left[P_{1,p}(t-h)-(1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})\right]$$

$$+\lambda h\left[1-\frac{\lambda}{\mu}\right]+\mu h P_{2,p}(t-h)$$

$$\frac{P_{1,p}(t)-P_{1,p}(t-h)}{h}=-\lambda_{\Sigma p}P_{1,p}(t-h)+\mu P_{2,p}(t-h)$$

as $h\to 0$

$$\frac{dP_{1,p}(t)}{dt}=-\lambda_{\Sigma p}P_{1,p}(t)+\mu P_{2,p}(t)$$

after initial conditions effects overcome:

$$\lambda_{\Sigma p}P_{1,p}=\mu P_{2,p}$$

For $n>1$, the steady state equation is the same as for the earlier case of exponential message length distribution and Poisson arrivals except that here for priority p substitute $\lambda_{\Sigma p}$ for λ

$$(\lambda_{\Sigma p}+\mu)P_{n,p}=\mu P_{(n+1),p}+\lambda_{\Sigma p}P_{(n-1),p} \qquad n>1$$

Assume as a general solution for $n\geq 1$

$$P_{n,p}=A\left(\frac{\lambda_{\Sigma p}}{\mu}\right)^n$$

Then since $$\sum_{n=0}^{\infty} P_{n,p}(t)=1$$

$$\left(1-\frac{\lambda}{\mu}\right)+A\sum_{n=1}^{\infty}\left(\frac{\lambda_{\Sigma p}}{\mu}\right)^n=1$$

$$A=\frac{\lambda}{\mu}\;\frac{\mu-\lambda_{\Sigma p}}{\lambda_{\Sigma p}}$$

$$P_{n,p}=\frac{\lambda}{\mu}\left[\frac{\mu-\lambda_{\Sigma p}}{\lambda_{\Sigma p}}\right]\left[\frac{\lambda_{\Sigma p}}{\mu}\right]^n \qquad n>0$$

$$P_0=\left(1-\frac{\lambda}{\mu}\right)$$

Multiplying $P_{n,p}$ and $C_{n,p}$ and summing to obtain the characteristic function of the waiting time

$$C_p=\left(1-\frac{\lambda}{\mu}\right)+\frac{\lambda}{\mu}\;\frac{\mu-\lambda_{\Sigma p}}{\lambda_{\Sigma p}}\sum_{n=1}^{\infty}\left[\frac{4\lambda_{\Sigma p}}{\left[(s+\gamma)+(s+\sigma)\right]^2}\right]^n$$

$$C_p=\left(1-\frac{\lambda}{\mu}\right)+\frac{2\lambda}{\mu}\left(\mu-\lambda_{\Sigma p}\right)\Big[\left(s+\lambda_{\Sigma p-1}+\mu-2\lambda_{\Sigma p}\right)+\left(s+\lambda_{\Sigma p-1}+\mu+2\sqrt{\mu\lambda_{\Sigma p-1}}\right)^{\frac{1}{2}}\left(s+\lambda_{\Sigma p-1}+\mu-2\sqrt{\lambda_{\Sigma p-1}\mu}\right)^{\frac{1}{2}}\Big]^{-1}$$

At this point it is convenient to convert time to units of

average message length and to let

$$\frac{\lambda}{\mu} = \rho \quad \lambda_{\Sigma p-1} = \alpha\lambda \quad \lambda_{\Sigma p} = \beta\lambda$$

then
$$\lambda = \rho \quad \lambda_{\Sigma p-1} = \alpha\rho \quad \lambda_{\Sigma p} = \beta\rho$$

The characteristic function then becomes

$$C_p = (1-\rho) + \frac{2\rho(1-\beta\rho)}{(s+\alpha\rho+1-2\beta\rho)+(s+\alpha\rho+1+2\sqrt{\alpha\rho})^{\frac{1}{2}}(s+\alpha\rho+1-2\sqrt{\alpha\rho})^{\frac{1}{2}}}$$

The inversion to the density function can be accomplished but results in an infinite sum of Bessel functions

$$w_p(t) = (1-\rho)\,\delta(t-0) + \frac{(1-\beta\rho)}{\beta} e^{-(\alpha\rho+1)t} \sum_{n=1}^{\infty} \left(\beta\sqrt{\frac{\rho}{\alpha}}\right)^n \frac{n\, I_n(2t\sqrt{\alpha\rho})}{t}$$

However the density function of the waiting time of the highest priority messages can be obtained by setting $\alpha = 0$. Then

$$C_1 = (1-\rho) + \frac{\rho(1-\beta\rho)}{(s+1-\beta\rho)}$$

$$w_1(t) = (1-\rho)\,\delta(t-0) + \rho(1-\beta\rho)e^{-(1-\beta\rho)t}$$

The distribution function of the waiting time of the highest priority messages

$$W_1(t) = \int_0^t w_1(t)\,dt = 1-\rho e^{-(1-\beta\rho)t}$$

Figures 1, 2 and 3 are plots of the waiting time distribution of highest priority messages for various values of ρ and β.

An approximate solution for $W_p(t)$ may be obtained by deriving the central moments of the waiting time distribution function from $\bar{C}_p$ and utilizing the Edgeworth Series[11] [12].

$$\nu_1 = \frac{\rho}{(1-\beta\rho)(1-\alpha\rho)}$$

$$\nu_2 = 2\rho\left[\frac{1}{(1-\beta\rho)^2(1-\alpha\rho)^2} + \frac{\alpha\rho}{(1-\beta\rho)(1-\alpha\rho)^3}\right]$$

$$\nu_3 = 6\rho\left[\frac{1}{(1-\beta\rho)^3(1-\alpha\rho)^3} + \frac{2\alpha\rho}{(1-\beta\rho)^2(1-\alpha\rho)^4} + \frac{\alpha\rho(1-\alpha\rho)}{(1-\beta\rho)(1-\chi\rho)^5}\right]$$

$$\nu_4 = 6\rho\left[\frac{4}{(1-\beta\rho)^4(1-\alpha\rho)^4} + \frac{12\alpha\rho}{(1-\beta\rho)^3(1-\alpha\rho)^5} + \frac{4\alpha\rho(2+3\alpha\rho)}{(1-\beta\rho)^2(1-\alpha\rho)^6}\right.$$

$$\left. + \frac{\alpha\rho\left[5(1+\alpha\rho)^2-(1-\alpha\rho)^2\right]}{(1-\beta\rho)(1-\chi\rho)^7}\right]$$

$$W_p(t) = F(x) = \Phi(x) - \frac{1}{3!}\frac{\mu_3}{\sigma^3}\varphi^{(2)}(x) + \frac{1}{4!}\left(\frac{\mu_4}{\sigma^4}-3\right)\varphi^{(3)}(x)$$

$$+ \frac{10}{6!}\left(\frac{\mu_3}{\sigma^3}\right)^2\varphi^{(5)}(x) + (1-\rho)$$

Where

$$x = \frac{t-\nu_1}{\sigma}$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\varphi^{(i)}(x) = \frac{d^i \varphi(x)}{dx^i}$$

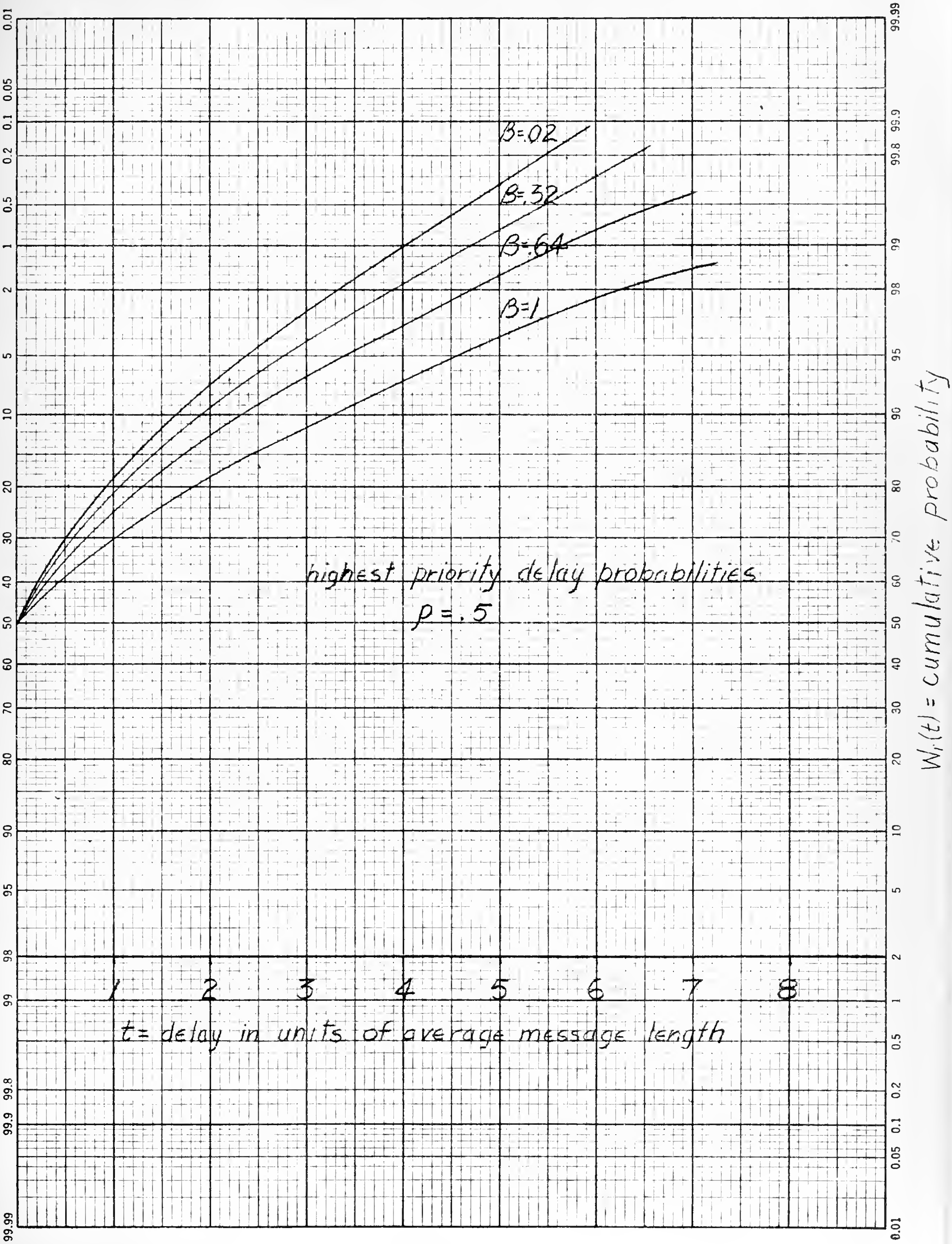
$$\Phi(x) = \int_{-\infty}^{x} \varphi(x)\,dx$$

$$\mu_2 = \sigma^2 = \nu_2 - \nu_1^2$$

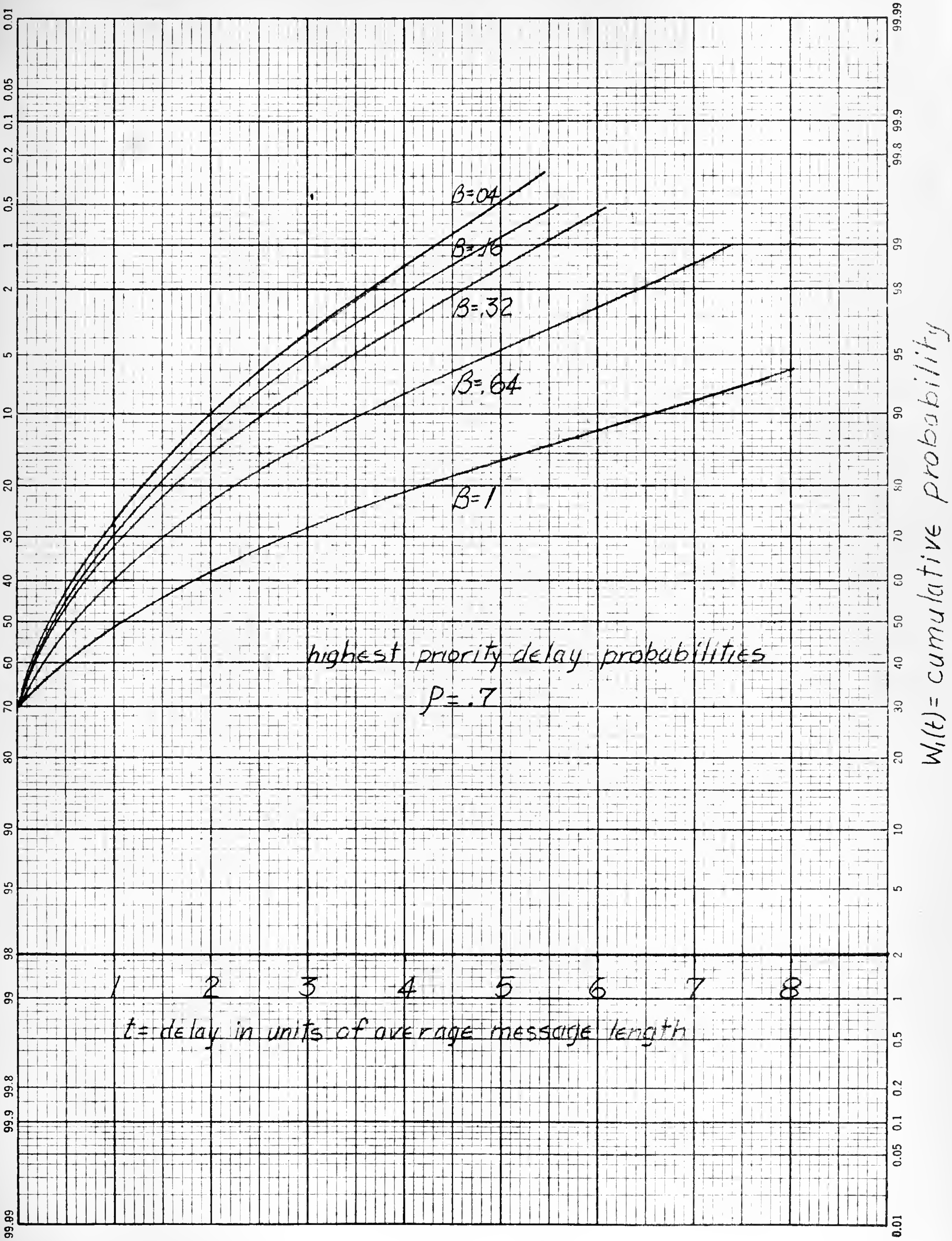
$$\mu_3 = \nu_3 - 3\nu_2 - 2\nu_1^3$$

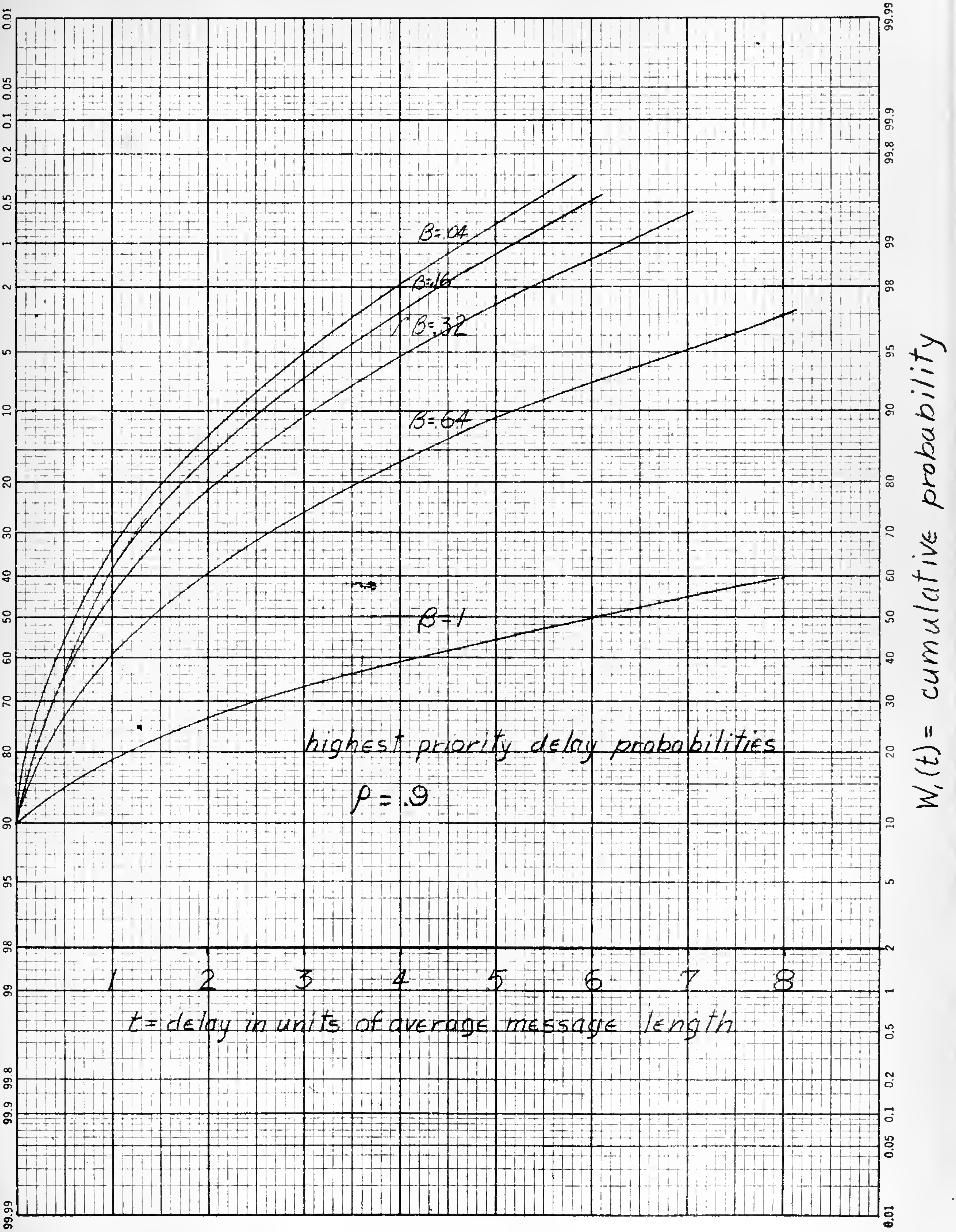
$$\mu_4 = \nu_4 - 4\nu_3\nu_1 + 6\nu_2\nu_1^2 - 3\nu_1^4$$

The effects of variations of α, β and ρ on the average waiting time and the standard deviation of the waiting time are demonstrated in Figures 4 and 5.

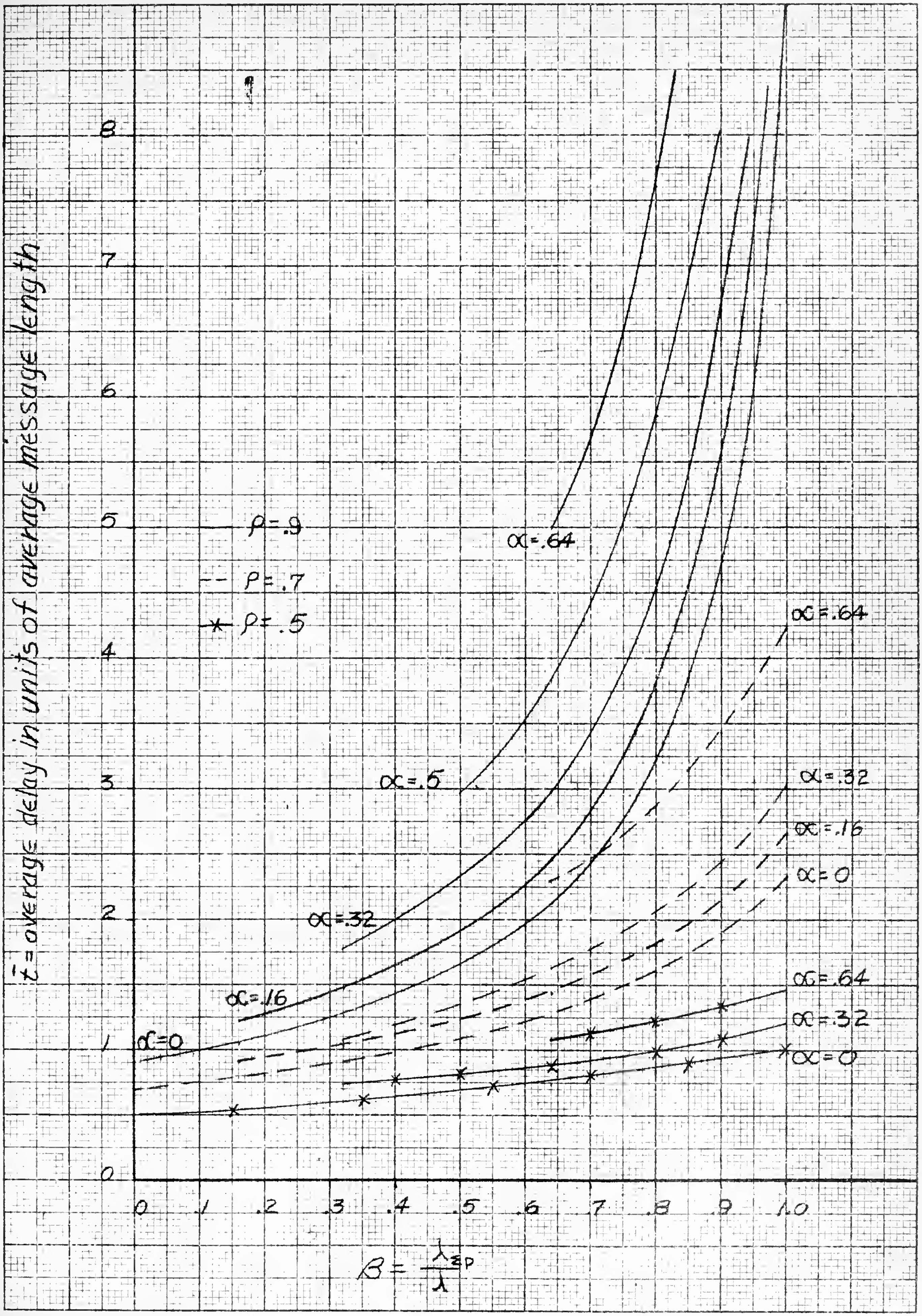


β=.02
β=.32
β=.64
β=1
highest priority delay probabilities
ρ=.5
t= delay in units of average message length
$W_1(t)$ = cumulative probability

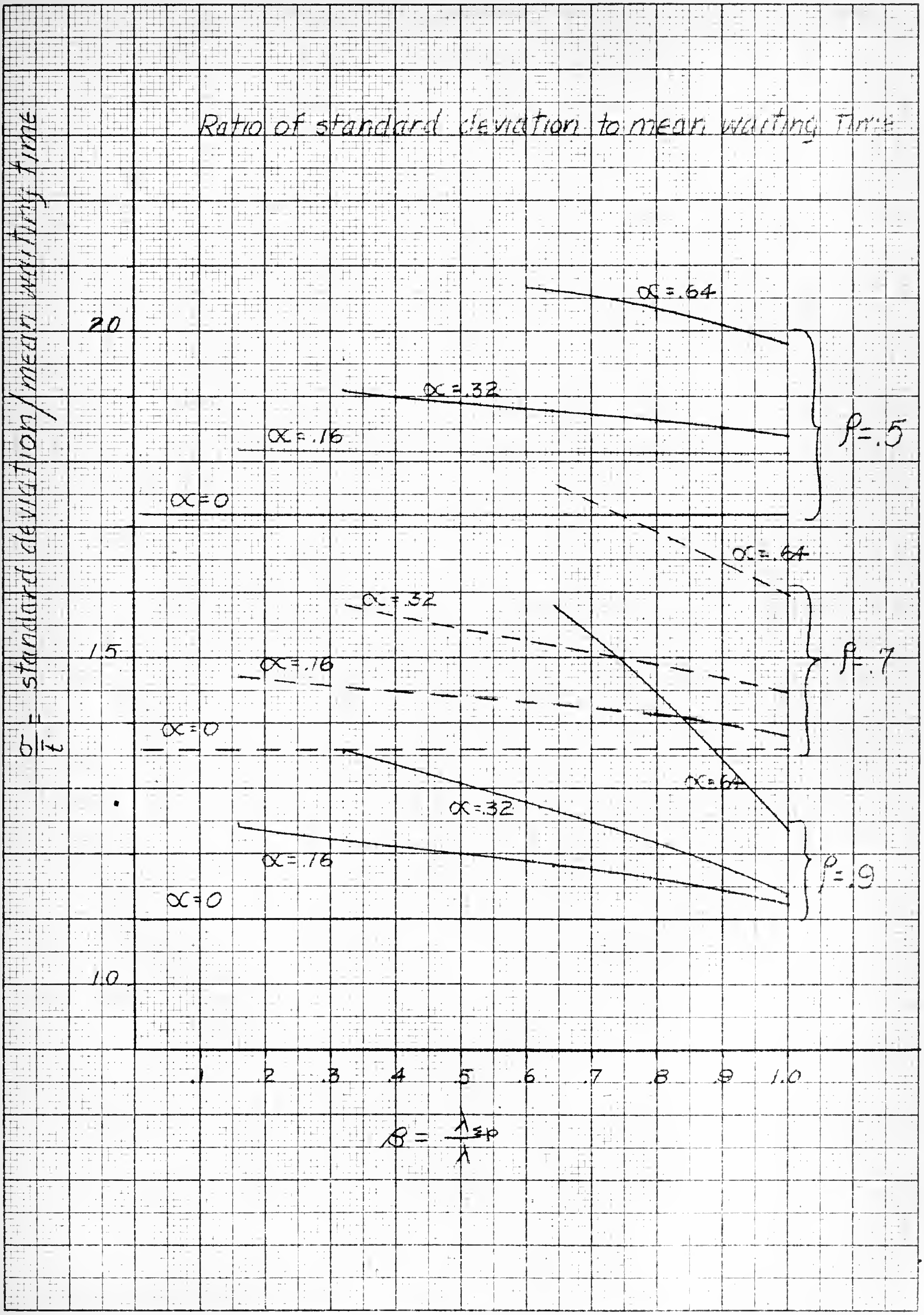

β=.04
β=.16
β=.32
β=.64
β=1
highest priority delay probabilities
ρ=.7
1 2 3 4 5 6 7 8
t= delay in units of average message length
W₁(t) = cumulative probability



highest priority delay probabilities
ρ = .9
β=.04
β=.16
β=.32
β=.64
β=1
t= delay in units of average message length
W₁(t)= cumulative probability
1
2
3
4
5
6
7
8



$\bar{t}$ = average delay in units of average message length
8
7
6
5
4
3
2
1
0
ρ = .9
ρ = .7
ρ = .5
α = .64
α = .5
α = .32
α = .16
α = 0
α = .64
α = .32
α = .16
α = 0
α = .64
α = .32
α = 0
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
$\beta = \frac{\lambda_{\Sigma P}}{\lambda}$



Ratio of standard deviation to mean waiting time
$\sigma/\bar{t}$ = standard deviation/mean waiting time
2.0
1.5
1.0
$\alpha=.64$
$\alpha=.32$
$\alpha=.16$
$\alpha=0$
$\rho=.5$
$\rho=.7$
$\rho=.9$
.1 .2 .3 .4 .5 .6 .7 .8 .9 1.0
$\beta = \frac{\lambda_{sp}}{\lambda}$

CHAPTER VI

CONCLUSIONS

1. Summary of results

In Chapter V expressions for the following were derived:

a. probabilities of the system states

b. the characteristic function of the waiting time distribution function

c. the probability density function of the waiting time distribution function

d. the waiting time distribution function of the highest priority messages

e. an approximate solution for the waiting time distribution function of messages of any priority.

These solutions are based on the following assumptions:

a. messages of each priority arrive in accordance with independent Poisson Laws

b. all priorities possess the same exponential message length distribution

c. the system consists of a single communication channel

d. standard military communication procedure is in effect.

2. Applications

In the light of present knowledge of the subject, the assumptions of Poisson arrivals and exponential message length distributions are considered satisfactory. The assumption concerning the equality of the mean message lengths of all priorities is not exact since in

general the higher the priority the shorter the message. However, if the lowest priority message length distribution is assumed for all priorities, the computed delays will be greater then those actually encountered and the system should perform at least as well as anticipated.

The military situation dictates the maximum message delay tolerable for each priority. The derived expressions enable a military communicator to analize a communication channel and determine if it satisfies the tactical requirements of the moment. For example the situation may require that at least 99% of the highest priority messages be delayed no longer than a particular time. Assuming that the assumptions are valid for the case considered, the communicator is able to determine how well the channel meets this requirement. Increasing the communication facilities, modifying the priority system, or further training of communications personnel (to decrease average transmission time) are indicated when requirements exceed channel capabilities.

Military communications is only one application of the results presented in Chapter V. Among others are aircraft landings, treatment of patients, and maintenance facilities. The various priorities may arise as a result of emergencies. Needless to say, it is necessary that the basic assumptions concerning the distributions of the arriving units be satisfied.

3. Further Investigation

A part of the original material presented in this paper resulted from an extension of an approach to the priority problem taken by

Alan Cobham [13] . It is hoped that this paper will be of assistance to future investigators who attempt to solve the more general problems in which the units of the various priorities possess different length distribution functions and the number of channels is greater than one.

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